

Integrability of Supergravity Billiards and the generalized Toda lattice equations[†]

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Abstract

We prove that the field equations of supergravity for purely time-dependent backgrounds, which reduce to those of a one-dimensional sigma model, admit a Lax pair representation and are fully integrable. In the case where the effective sigma model is on a maximally split non-compact coset U/H (maximal supergravity or subsectors of lower supersymmetry supergravities) we are also able to construct a completely explicit analytic integration algorithm, adapting a method introduced by Kodama et al in a recent paper. The properties of the general integral are particularly suggestive. Initial data are represented by a pair \mathcal{C}_0, h_0 where \mathcal{C}_0 is in the CSA of the Lie algebra of U and $h_0 \in H/\mathcal{W}$, is in the compact subgroup H modded by the Weyl group of U . At asymptotically early and asymptotically late times the Lax operator is always in the Cartan subalgebra and due to the isospectral property the two limits differ only by the action of some element of the Weyl group. Hence the entire cosmic evolution can be seen as a billiard scattering with quantized angles defined by the Weyl group. The solution algorithm realizes a map from H/\mathcal{W} into \mathcal{W} .

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1 Introduction

String Cosmology The cosmological implications of superstring theory have been under attentive consideration in the last few years from various viewpoints [1]. This involves the classification and the study of possible time-evolving string backgrounds which amounts to the construction, classification and analysis of supergravity solutions depending only on time or, more generally, on a low number of coordinates including time.

The idea of cosmic billiards In this context a quite challenging and potentially highly relevant phenomenon for the overall interpretation of extra-dimensions and string dynamics is provided by the so named *cosmic billiard* phenomenon [2],[3],[4], [5]. This is based on the relation between the cosmological scale factors and the duality groups U of string theory. The group U appears as isometry group of the scalar manifold \mathcal{M}_{scalar} emerging in compactifications of 10-dimensional supergravity to lower dimensions $D < 10$ and depends both on the geometry of the compact dimensions and on the number of preserved supersymmetries $N_Q \leq 32$. For $N_Q > 8$ the scalar manifold is always a homogeneous space U/H . The cosmological scale factors $a_i(t)$ associated with the various dimensions of supergravity are interpreted as exponentials of those scalar fields $h_i(t)$ which lie in the Cartan subalgebra of \mathbb{U} , while the other scalar fields in U/H correspond to positive roots $\alpha > 0$ of the Lie algebra \mathbb{U} . The cosmological evolution is described by a *fictitious ball* that moves in the CSA of \mathbb{U} and occasionally bounces on the hyperplanes orthogonal to the various roots: the billiard walls. Such bounces represent inversions in the time evolution of scale factors. Such a scenario was introduced by Damour, Henneaux, Julia and Nicolai in a series of papers that we already quoted [2],[3], [4],[5], which generalize classical results obtained in the context of pure General Relativity [6]. In this approach the cosmic billiard phenomenon is analyzed as an asymptotic regime in the neighborhood of space-like singularities and the billiard walls are seen as delta function potentials provided by the various p -forms of supergravity localized at sharp instants of time.

The smooth cosmic billiard programme In a series of papers [7],[8, 9, 10] involving both the present authors and other collaborators it was started and developed what can be described as the *smooth cosmic billiard programme*. This amounts to the study of the *billiard features*, namely inversions in the evolution of cosmological scale factors within the framework of exact analytic solutions of supergravity rather than in asymptotic regimes. Crucial starting point in this programme was the observation [7] that the fundamental mathematical setup underlying the appearance of the billiard phenomenon is the so named *Solvable Lie algebra parametrization* of supergravity scalar manifolds, pioneered in [11] and later applied to the solution of a large variety of superstring/supergravity problems [12],[13],[14] (for a comprehensive review see [15]). Thanks to the solvable parametrization, one can establish a precise algorithm to implement the following programme:

1. Reduce the original supergravity in higher dimensions $D \geq 4$ (for instance $D = 10, 11$) to a gravity-coupled σ -model in $D \leq 3$ where gravity is non-dynamical and can be eliminated. The target manifold is the non compact coset $U/H \cong \exp[Solv(U/H)]$ metrically equivalent to a solvable group manifold.

2. Utilize the structure of the algebra $Solv(U/H)$ in order to integrate analytically the σ -model equations.
3. Dimensionally oxidize the solutions obtained in this way to exact time dependent solutions of $D \geq 4$ supergravity.

Within this approach it was proved in [7] that the *cosmic billiard phenomenon* is indeed a general feature of exact time dependent solutions of supergravity and has *smooth realizations*. Calling $\mathbf{h}(t)$ the r -component vector of Cartan fields (where r is the rank of U) and $\mathbf{h}_\alpha(t) \equiv \alpha \cdot \mathbf{h}(t)$ its projection along any positive root α , a *bounce* occurs at those instant of times t_i such that:

$$\exists \alpha \in \Delta_+ \quad \setminus \quad \dot{\mathbf{h}}_\alpha(t) |_{t=t_i} = 0 \quad (1.1)$$

namely when the Cartan field in the direction of some root α inverts its behaviour. All higher dimensional bosonic fields (off-diagonal components of the metric $g_{\mu\nu}$ or p -forms $A^{[p]}$) are, via the solvable parametrization of U/H , in one-to-one correspondence with roots $\phi_\alpha \Leftrightarrow \alpha$, and each bounce on a hyperplane orthogonal to a root α is caused by the sudden growing of the field ϕ_α which, in exact smooth solutions, has a typical bell-shaped behaviour around the bouncing time $t = t_b$.

1.1 Algebraic structure of U/H and integration with the compensator method

As alluded to in the above discussion, the actual construction of exact analytic solutions performed in [7] and [8] was based on two ingredients, namely the systematics of *dimensional reduction/oxidation* which ultimately allows to reduce supergravity field equations to those of a one-dimensional sigma model on a coset U/H and secondly the algebraic structure of U/H which allows to devise effective integration methods. The essential point relative to the second ingredient is that in all supergravity models U/H is non-compact and for maximal supersymmetry it is also *maximally split*, namely the Lie algebra U of U is the maximally non compact real section of its own complexification and H is generated by the maximal compact subalgebra $\mathbb{H} \subset U$. In this case the generators t_α of \mathbb{H} are in one-to-one correspondence with the positive roots α of U and the detailed algebraic structure of this geometry was utilized in [7] to devise an integration algorithm named *the compensator method*. This latter essentially consists of substituting the original σ -model field equations with an equivalent set of differential equations which are more manageable since they can be put into triangular form and integrated one-by-one. The geometric meaning of these new equations is that they correspond to the conditions to be satisfied by the $\theta_\alpha(t)$ parameters of an H -gauge transformation $\mathbb{L} \mapsto \mathbb{L}h(\theta)$ in order for the so called solvable gauge of the U/H coset representative $\mathbb{L}(t)$ to be preserved, solvable gauge being the possibility of writing $\mathbb{L}(t)$ as an exponential of the solvable Lie algebra $Solv(U/H)$. The limitations of the approach developed in [7, 8] and of the compensator method which nonetheless was very valuable in producing a rich set of explicit analytic solutions are threefold:

- a** In its original form, the scheme of [7, 8] applies only to *finite algebras*, namely to the dimensional reduction of supergravity to $D = 3$, which is the first where all (Bose) degrees of freedom can be represented by scalar fields. This is not yet sufficient to reveal the full algebraic structure underlying time evolution and billiard dynamics. Indeed to discuss this latter one can still dimensionally reduce to $D = 2$ and $D = 1$ where the algebra \mathbb{U} is extended to an affine or hyperbolic Kač–Moody algebra. It is therefore mandatory to consider the field theoretical mechanisms of such Kač–Moody extensions and subsequently extend the so far considered algebraically based integration methods to Kač–Moody algebras.
- b** Heavily relying on the *maximally split* character of the pair $\{\mathbb{U}, \mathbb{H}\}$ the original form of the scheme presented in [7, 8] is not directly applicable to the case of lower supersymmetry $N_Q < 32$ and hence to more realistic and more interesting compactifications than pure toroidal ones.
- c** The compensator method, although very valuable in obtaining explicit solutions, replaces one differential system with another one and therefore does not answer the question whether the system is a fully integrable system nor it is able to provide a uniformly given, general integral, depending on as many integration constants as necessary. Indeed the solution strategy is based on triangulization of the set of equations and the choice of parametrization of the H-subgroup element h (order of the rotations $\exp[\theta_\alpha(t)t_\alpha]$) in such a way that the resulting differential system for the rotation angles $\theta_\alpha(t)$ be triangular is something that has to be decided case by case and both by intuition and by trial and error.

The two recent papers [10] and [9] have presented some substantial advances with respect to points **a]** and **b]** of the above list.

a]: In [10] it was clarified the general field theoretical mechanism underlying Kač–Moody extensions in supergravity and the general pattern of such extensions in the reduction to $D = 2$ of all $D = 4$ supergravity theories with all number of supercharges N_Q was also presented. This means that such extensions are now under control for all compactifications of both M-theory and string theory on all type of compact geometries. Indeed these latter lead to the various different types of $D = 4$ supergravities. Although integration methods for the Kač–Moody analogue of the σ –model equations were not presented in [10], yet the results there obtained provide the necessary prerequisite in the development of such a programme. Indeed the field theoretical mechanism of the extension is what one needs to consider in order to rewrite the actual $D = 2$ or $D = 1$ field equations in a manifestly Kač–Moody covariant way.

b]: In [9], instead, exploiting in a systematic way the Tits Satake theory of real forms [16] for simple algebras and the Tits-Satake projection of root-spaces, a decisive advance was made with respect to point **b]** of the above list. Indeed it was shown how the differential equations of the σ –model on a *non maximally split non compact coset* \mathbb{U}/\mathbb{H} , typically emerging from non maximal supergravities, can be consistently reduced to those

of a σ -model on a *maximally split one*: U_{TS}/H_{TS} . By $U_{TS} \subset U$ we denote the Tits-Satake subalgebra of the original algebra U which is uniquely defined by the involutive automorphism σ associated with the real form of U . By $\mathbb{H}_{TS} \subset U_{TS}$ we denote the maximal compact subalgebra of the Tits Satake subalgebra which, as in all maximally split cases, has a number of generators equal to the number of positive roots in the $\overline{\Delta}_U$ root system of U_{TS} . This latter is the Tits Satake projection of the U -root system:

$$\Pi : \Delta_U \mapsto \overline{\Delta}_U \quad (1.2)$$

obtained by setting to zero all transverse components α_\perp of each root α . Indeed the automorphism $\sigma^2 = 1$ defining the real form of U splits the Cartan subalgebra $CSA \subset U$ in two eigenspaces $\mathcal{H}^{n.c.}$ and \mathcal{H}^{comp} , respectively corresponding to the eigenvalues ± 1 . Each root α , which is a linear functional on the CSA, is accordingly decomposed into a parallel and into a transverse part to the non compact Cartan subalgebra $\mathcal{H}^{n.c.}$:

$$\alpha = \alpha_\parallel \oplus \alpha_\perp \quad ; \quad \alpha_\perp(\mathcal{H}^{n.c.}) = 0 \quad ; \quad \alpha_\parallel(\mathcal{H}^{n.c.}) \neq 0. \quad (1.3)$$

The Tits Satake projection projects each root α onto its parallel part α_\parallel .

As already mentioned it was shown in [9] that in the *non maximally split cases* one can consistently reduce the σ -model equations of U/H to those of U_{TS}/H_{TS} and furthermore it was proved that one can lift back any solution of these latter to a full fledged solution of the original system by means of the action of an automorphism compact group named the paint group $G_{paint} \subset U$ with respect to which the entire solvable algebra $Solv_U$ transforms as a representation:

$$[G_{paint}, Solv_U] = Solv_U. \quad (1.4)$$

Hence a large class of solutions of the σ -model on *non-maximally split coset manifolds*, and presumably the essentially relevant class for billiard dynamics, since it contains all the Cartan fields and all the walls on which the cosmic ball can bounce, can be obtained from the general solution of σ -models on *maximally split cosets*.

1.2 The present paper solves point c]: full integrability via LAX representation

In the present paper we shall present a complete solution for the third point in the above list of problems and required generalizations. Indeed we shall prove that the one-dimensional σ -model on *maximally split cosets* is completely integrable in the sense of integrable systems, since the first order equations for the tangent vector to a geodesic can be recast in the classical form of a LAX system. Furthermore, adapting to our case an algorithm developed in [17], we can write a closed analytic form for the general integral, depending on a complete set of integration constants.

This new algorithm replaces the compensator method and does not require the solution of neither differential nor algebraic equations. The solution of the equations for the time parameter ranging from $-\infty$ to $+\infty$ is directly constructed in terms of the initial data fixed say at $t = 0$. It is particularly interesting to study the general properties of the solution algorithm. As we shall explain, the initial data parametrizing the value L_0 of the Lax operator $L(t)$ at $t = 0$ can be represented as a pair:

1. An element of the Cartan subalgebra $\mathcal{C}_0 \in \text{CSA} \subset \mathbb{U}$.
2. An element $h_0 \in \text{H} = \exp[\mathbb{H}]$ of the maximally compact subgroup $\text{H} \subset \text{U}$.

The solution algorithm \mathcal{S}_K provides a general integral

$$\mathcal{S}_K : \{\mathcal{C}_0, h_0\} \implies L(t | \mathcal{C}_0, h_0) \quad (1.5)$$

with the following properties:

1. $\forall w \in \mathcal{W}$ where $\mathcal{W} \subset \text{H} \subset \text{U}$ is the Weyl group of the Lie algebra \mathbb{U}

$$L(t | \mathcal{C}_0, w \cdot h_0) = L(t | w^{-1} \cdot \mathcal{C}_0 \cdot w, h_0). \quad (1.6)$$

2. The limits of the Lax operator $L(t | \mathcal{C}_0, h_0)$ at $t = \pm\infty$ lie in the Cartan subalgebra:

$$\lim_{t \rightarrow \pm\infty} L(t | \mathcal{C}_0, h_0) = L_{\pm\infty}(\mathcal{C}_0, h_0) \in \text{CSA}. \quad (1.7)$$

3. At any instant of time the eigenvalues $\{\lambda_1 \lambda_2 \dots \lambda_n\}$ of the Lax operator are the same and are given by

$$\lambda_i = \mathbf{w}_i(\mathcal{C}_0) \quad (1.8)$$

where \mathbf{w}_i are the weights of the representation of \mathbb{U} to which the Lax operator is assigned.

Property (2) and property (3) combined together imply that the two asymptotic values $L_{\pm\infty}(\mathcal{C}_0, h_0)$ of the Lax operator are necessarily related to each other by some element $\Omega \in \mathcal{W}$ of the Weyl group which represents a sort of topological charge of the solution:

$$\exists \Omega \in \mathcal{W} : L_{+\infty}(\mathcal{C}_0, h_0) = \Omega^{-1} \cdot L_{-\infty}(\mathcal{C}_0, h_0) \cdot \Omega. \quad (1.9)$$

Indeed since the Weyl group is discrete, by varying the initial datum h_0 a solution which is characterized by a *charge* Ω_1 , cannot continuously be deformed into another characterized by a different *charge* Ω_2 .

It happens furthermore that the topological charge Ω of a solution is independent of the choice of $\mathcal{C}_0 \in \text{CSA}$ and just depends on h_0 . We conclude that the solution algorithm \mathcal{S}_K developed in [17] induces a map:

$$\mathcal{P}_K : \frac{\text{H}}{\mathcal{W}} \rightarrow \mathcal{W} \equiv \pi_1\left(\frac{\text{H}}{\mathcal{W}}\right) \quad (1.10)$$

from the non-simply connected coset manifold $\frac{\text{H}}{\mathcal{W}}$ to its own homotopy group.

Let us now proceed to the derivation of these results and to the illustration of their application to cosmic billiards.

2 One-dimensional sigma models on maximally split coset manifolds and Lax pairs

In view of the discussion presented in the introduction our mathematical object of study is reduced to be the one-dimensional σ -model on a maximally split hermitian symmetric coset manifold U/H . The corresponding action principle is given by:

$$S = \int dt \, h_{IJ}(\phi) \frac{d\phi^I}{dt} \frac{d\phi^J}{dt} \quad (2.1)$$

where h_{IJ} is the canonical metric on the coset, which we parameterize by scalar fields named ϕ^I . By definition, the coset U/H is hermitian symmetric if the Lie algebra \mathbb{U} of the group U and the subalgebra \mathbb{H} of the subgroup H fulfill the relations:

$$[\mathbb{H}, \mathbb{H}] \subset \mathbb{H}, \quad [\mathbb{H}, \mathbb{K}] \subset \mathbb{K}, \quad [\mathbb{K}, \mathbb{K}] \subset \mathbb{H} \quad (2.2)$$

having called \mathbb{K} the orthogonal complement of \mathbb{H} in \mathbb{U} , namely:

$$\mathbb{H} \subset \mathbb{U}, \quad \mathbb{U} = \mathbb{H} \oplus \mathbb{K}. \quad (2.3)$$

In order to asset the further property of being maximally split we assume that \mathbb{U} is a *simple Lie algebra* in its maximally non-compact real section and we normalize the Cartan–Weyl commutation relation in the standard form:

$$\begin{aligned} [\mathcal{H}_i, \mathcal{H}_j] &= 0, \\ [\mathcal{H}_i, E^\alpha] &= \alpha^i E^\alpha, \\ [E^\alpha, E^{-\alpha}] &= \alpha \cdot \mathcal{H}, \\ [E^{\alpha_i}, E^{\alpha_j}] &= \mathcal{N}_{\alpha_i, \alpha_j} E^{\alpha_i + \alpha_j} \end{aligned} \quad (2.4)$$

where \mathcal{H}_i denotes a basis for the Cartan subalgebra (CSA), α_i the root vectors of the root system Δ and E^{α_i} the corresponding step operators. Then U/H is maximally split if the subalgebra \mathbb{H} is chosen to be the maximally compact subalgebra which has dimension equal to the number of positive roots and which can be defined as:

$$\mathbb{H} = \text{Span}\{t_\alpha\} = \text{Span}\{(E^\alpha - E^{-\alpha})\}. \quad (2.5)$$

Eq. (2.5) defines the subalgebra and also a basis for it given by the generators $t_\alpha \equiv (E^\alpha - E^{-\alpha})$. In this case the complementary space \mathbb{K} is given by

$$\mathbb{K} = \text{Span}\{\mathbb{K}_A\} = \text{Span}\left\{\mathcal{H}_i, \frac{1}{\sqrt{2}}(E^\alpha + E^{-\alpha})\right\} \quad (2.6)$$

which, similarly to eq.(2.5) introduces a canonical basis of generators \mathbb{K}_A .

For any parametrization of the coset provided by a coset representative $\mathbb{L}(\phi)$ one can write down the \mathbb{U} -algebra valued left invariant one-form

$$\Omega = \mathbb{L}^{-1} \frac{d}{dt} \mathbb{L} = W^\alpha t_\alpha + V^A \mathbb{K}_A \equiv W + V \quad (2.7)$$

where V is the coset manifold vielbein using which one can rewrite action (2.1) as follows

$$S = \int dt \operatorname{Tr} (VV). \quad (2.8)$$

In eq. (2.8) it is understood that we have chosen a linear representation of the Lie algebra \mathbb{U} (which one is irrelevant, typically the lowest dimensional one) and therefore the vielbein has become a $n \times n$ matrix valued one-form, n being the dimension of the chosen representation:

$$V(t) = \mathbb{K}_A V^A_I(\phi(t)) \frac{d\phi^I}{dt} = \mathbb{K}_A V^A. \quad (2.9)$$

Our goal is to solve the classical field equation of the model (2.1), namely calculate the geodesics of the manifold \mathbb{U}/\mathbb{H} . Let us calculate the variation of the action

$$\delta S = 2 \int dt \operatorname{Tr} (V\delta V). \quad (2.10)$$

In order to derive a convenient expression for the variation of the vielbein δV which enters eq. (2.10), we represent the variation of the left-invariant one-form Ω defined in eq. (2.7) in the following equivalent way:

$$\delta\Omega = [\Omega, \mathbb{L}^{-1}\delta\mathbb{L}] + \frac{d}{dt}(\mathbb{L}^{-1}\delta\mathbb{L}) \quad (2.11)$$

and take into account that

$$\mathbb{L}^{-1}\delta\mathbb{L} = \delta\omega^\alpha t_\alpha + \delta v^A \mathbb{K}_A \equiv \delta\omega + \delta v, \quad (2.12)$$

where δv^I and $\delta\omega^\alpha$ are functionals of the variation of the fields $\delta\phi^I$, and there is the one-to-one correspondence between δv^I and $\delta\phi^I$ (and between v^I and ϕ^I modulo trivial constants), so that the transition from $\delta\phi^I$ to δv^I can be treated as a change of the basis. Substituting the relations (2.7) and (2.12) into eq. (2.11) we obtain the following new relation:

$$\delta W + \delta V = [W + V, \delta\omega + \delta v] + \frac{d}{dt}(\delta\omega + \delta v). \quad (2.13)$$

Now, using eq. (2.2) it is easy to extract from eq. (2.13) the projection along the coset generators \mathbb{K}_A and derive δV

$$\delta V = [W, \delta v] + [V, \delta\omega] + \frac{d}{dt}\delta v. \quad (2.14)$$

Substituting this expression into (2.10) one can easily obtain the variation of the action δS

$$\begin{aligned} \delta S &= 2 \int dt \operatorname{Tr} \left(V[W, \delta v] + V[V, \delta\omega] + V \frac{d}{dt}\delta v \right) \\ &= 2 \int dt \operatorname{Tr} \left(\left([V, W] - \frac{d}{dt}V \right) \delta v \right). \end{aligned} \quad (2.15)$$

The equations of motion follow immediately from the vanishing of the action variation $\delta S = 0$

$$\frac{d}{dt}V = [V, W] \quad (2.16)$$

in the form of a Lax pair representation. Hence the resulting system is indeed integrable. The direct consequence of the Lax pair representation (2.16) is the existence of the following integrals of motion:

$$\mathcal{I}_k = \text{Tr} (V^k) . \quad (2.17)$$

for all integers $k \leq n$, having called n the dimension of the matrix V namely the dimension of the chosen linear representation of the Lie algebra \mathbb{U} , as we have already specified.

2.1 Nomizu connection for maximally split cosets and the Lax pair representation

Now in order to make contact with our previous work and with the discussion of cosmic billiards, we want to compare the Lax pair representation of the field equations (2.16) with the form of the same equations derived in [7] by means of the Nomizu connection on solvable Lie algebras. There showed that the field equations of the purely time dependent σ -model (2.1), which is what we are supposed to solve in our quest for time dependent solutions of supergravity, can be written as follows:

$$\dot{Y}^A + \Gamma_{BC}^A Y^B Y^C = 0 \quad (2.18)$$

where Y^A denotes the purely time dependent tangent vectors to the geodesic in an anholonomic basis:

$$Y^A = \begin{cases} Y^i = V_I^i(\phi) \dot{\phi}^I & i \in \text{CSA} \\ Y^\alpha = \sqrt{2} V_I^\alpha(\phi) \dot{\phi}^I & \alpha \in \text{positive root system } \Delta_+ \end{cases} \quad (2.19)$$

$V_I^A(\phi) d\phi^I$ being the vielbein of the target manifold we are considering, namely the same object already introduced in the previous section. In eq. (2.18) the symbol Γ_{BC}^A denotes the components of the Levi-Civita connection in the chosen anholonomic basis. Explicitly they are related to the components of the Levi-Civita connection in an arbitrary holonomic basis by:

$$\Gamma_{BC}^A = \Gamma_{JK}^I V_I^A V_B^J V_C^K - \partial_K (V_J^A) V_B^J V_C^K \quad (2.20)$$

where the inverse vielbein is defined in the usual way:

$$V_I^A V_B^I = \delta_B^A . \quad (2.21)$$

The basic idea of [7], which was exploited together with the *compensator method* in order to construct explicit solutions, is the following. The connection Γ_{BC}^A can be identified with the *Nomizu connection* defined on a solvable Lie algebra, if the coset representative \mathbb{L} from which we construct the vielbein is solvable, namely if it is represented as the

exponential of the associated solvable Lie algebra $Solv(U/H)$. In fact, as we can read in [18] once we have defined over $Solv$ a non degenerate, positive definite symmetric form:

$$\begin{aligned} \langle , \rangle & : Solv \otimes Solv \longrightarrow \mathbb{R}, \\ \langle X, Y \rangle & = \langle Y, X \rangle \end{aligned} \quad (2.22)$$

whose lifting to the manifold produces the metric, the covariant derivative is defined through the **Nomizu operator**:

$$\forall X \in Solv : \mathbb{L}_X : Solv \longrightarrow Solv \quad (2.23)$$

so that

$$\forall X, Y, Z \in Solv : 2\langle Z, \mathbb{L}_X Y \rangle = \langle Z, [X, Y] \rangle - \langle X, [Y, Z] \rangle - \langle Y, [X, Z] \rangle \quad (2.24)$$

while the Riemann curvature 2-form is given by the commutator of two Nomizu operators:

$$R^W_Z(X, Y) = \langle W, \{[\mathbb{L}_X, \mathbb{L}_Y] - \mathbb{L}_{[X, Y]}\} Z \rangle. \quad (2.25)$$

This implies that the covariant derivative explicitly reads:

$$\mathbb{L}_X Y = \Gamma_{XY}^Z Z \quad (2.26)$$

where

$$\Gamma_{XY}^Z = \frac{1}{2} (\langle Z, [X, Y] \rangle - \langle X, [Y, Z] \rangle - \langle Y, [X, Z] \rangle) \frac{1}{\langle Z, Z \rangle} \quad \forall X, Y, Z \in Solv. \quad (2.27)$$

Eq.(2.27) is true for any solvable Lie algebra, but in the case of **maximally non-compact, split algebras** we can write a general form for Γ_{XY}^Z , namely:

$$\begin{aligned} \Gamma_{jk}^i &= 0, \\ \Gamma_{\alpha\beta}^i &= \frac{1}{2} (-\langle E_\alpha, [E_\beta, H^i] \rangle - \langle E_\beta, [E_\alpha, H^i] \rangle) = \frac{1}{2} \alpha^i \delta_{\alpha\beta}, \\ \Gamma_{ij}^\alpha &= \Gamma_{i\beta}^\alpha = \Gamma_{j\alpha}^i = 0, \\ \Gamma_{\beta i}^\alpha &= \frac{1}{2} (\langle E^\alpha, [E_\beta, H_i] \rangle - \langle E_\beta, [H_i, E^\alpha] \rangle) = -\alpha_i \delta_\beta^\alpha, \\ \Gamma_{\alpha\beta}^{\alpha+\beta} &= -\Gamma_{\beta\alpha}^{\alpha+\beta} = \frac{1}{2} N_{\alpha\beta}, \\ \Gamma_{\alpha+\beta\beta}^\alpha &= \Gamma_{\beta\alpha+\beta}^\alpha = \frac{1}{2} N_{\alpha\beta} \end{aligned} \quad (2.28)$$

where $N^{\alpha\beta}$ is defined by the commutator $[E_\alpha, E_\beta] = N_{\alpha\beta} E_{\alpha+\beta}$, as usual. The explicit form (2.28) follows from the choice of the non degenerate metric:

$$\begin{aligned} \langle \mathcal{H}_i, \mathcal{H}_j \rangle &= 2\delta_{ij}, \\ \langle \mathcal{H}_i, E_\alpha \rangle &= 0, \\ \langle E_\alpha, E_\beta \rangle &= \delta_{\alpha,\beta} \end{aligned} \quad (2.29)$$

$\forall \mathcal{H}_i, \mathcal{H}_j \in \text{CSA}$ and $\forall E_\alpha$, step operator associated with a positive root $\alpha \in \Delta_+$.

Hence in the case of maximally split algebras the first order equations, take the general form:

$$\begin{aligned} \dot{Y}^i &+ \frac{1}{2} \sum_{\alpha \in \Delta_+} \alpha^i Y_\alpha^2 = 0, \\ \dot{Y}^\alpha &+ \sum_{\beta \in \Delta_+} N_{\alpha\beta} Y^\beta Y^{\alpha+\beta} - \alpha_i Y^i Y^\alpha = 0 \end{aligned} \quad (2.30)$$

which follows from eq. (2.28).

Our next point is that equations (2.30) are identical to the Lax pair equations (2.16) if we identify:

$$\begin{aligned} V &= V^A \mathbb{K}_A \equiv Y^i \mathcal{H}_i + Y^\alpha \frac{1}{2} (E^\alpha + E^{-\alpha}), \\ W &= W^\alpha \mathbb{H}_\alpha \equiv Y^\alpha \frac{1}{2} (E^\alpha - E^{-\alpha}). \end{aligned} \quad (2.31)$$

On one hand insertion of (2.31) into (2.16), upon use of the standard Cartan Weyl commutation relations (2.4), yields (2.30), on the other hand eqs. (2.31) are justified as follows. From eq.(2.19) we have

$$V^\alpha = \frac{1}{\sqrt{2}} Y^\alpha \quad (2.32)$$

while, on the other hand we have that the condition for the coset representative $\mathbb{L}(\phi)$ to be solvable is given by:

$$V^\alpha = \sqrt{2} W^\alpha. \quad (2.33)$$

Together eqs. (2.32) and (2.33) imply (2.31). In this way we have proved that the first order equations (2.30) studied in [7, 8, 9] and for which we have also recently found bouncing solutions [9] also in the interesting case of $F_{4(4)}/\text{Usp}(6) \times \text{SU}(2)$, which is relevant for $N = 6$ supergravity, are actually fully integrable since they admit the Lax pair rewriting (2.16). In [7, 8, 9] eqs. (2.30) were addressed by means of the compensator method. Let us recall its form in order to compare it with the general integral formula provided by the algorithm of [17].

2.2 Minireview of the compensator method

As we have seen above, the condition which ensure that a coset representative $\mathbb{L}(\phi)$ is solvable, namely that can be represented as the exponential of the solvable Lie algebra $\text{Solv}(\text{U}/\text{H})$ is given by the linear relation (2.33) on the components of the left-invariant one-form (2.7). The compensator method streams from the following argument. Given a solvable coset representative $\mathbb{L}(\phi)$, one asks the question how many other solvable representatives are there in the same equivalence class. This amounts to investigating the condition to be satisfied by the H-gauge transformation

$$\begin{aligned} \mathbb{L} &\mapsto \mathbb{L} h = \overline{\mathbb{L}}, \\ h &= \exp [\theta^\alpha t_\alpha] \end{aligned} \quad (2.34)$$

in order for the solvable gauge (2.33) to be preserved. This condition is a system of differential equations, namely:

$$\frac{\sqrt{2}}{\text{tr}(t_\alpha^2)} \text{tr} (h^{-1}(\theta) dh(\theta) t_\alpha) = V^\beta \left(-A(\theta)_\beta^\alpha + D(\theta)_\beta^\alpha \right) + V^i D(\theta)_i^\alpha. \quad (2.35)$$

In eq. (2.35) the matrix $A(\theta)$ is the adjoint representation of $h \in \mathbf{H}$ and $D(\theta)$ is the D -representation of the same group element which acts on the complementary space \mathbb{K} and which depends case to case

$$\begin{aligned} h^{-1} t_\alpha h &= A(\theta)_\alpha^\beta t_\beta, \\ h^{-1} \mathbb{K}_A h &= D(\theta)_A^B \mathbb{K}_B. \end{aligned} \quad (2.36)$$

The differential system (2.35) is actually equivalent to the original system (2.30). To see this it suffices to argue as follows. A simple solution of the first order equations (2.30) is easily obtained by setting $Y^\alpha = 0$ and $Y^i = c^i = \text{const}$, namely we can begin with a constant vector in the direction of the CSA. Such a solution is named the *normal form* of the tangent vector. In the language of billiard dynamics it corresponds to a *fictitious ball* that moves on a straight line with a constant velocity. All other solutions of eqs. (2.30) can be obtained from the normal form solution by means of successive rotations of the compact group, with parameters $\theta[t]$ satisfying the differential equation (2.35). The advantage of this method, emphasized in [7] where we introduced it is that we can choose rotations in such a way that at each successive rotation we obtain an equation which is fully integrable in terms of the integral of the previous ones. How to do this, however, can not be prescribed in general and furthermore it is not obvious how many steps one can do of this type. In other words, although the compensator method provides a valuable tool to obtain several explicit solutions, it does not suffice to reveal the complete integrability which is instead revealed by the Lax pair representation (2.16).

2.3 The A_2 case with the compensator method

To appreciate the bearing of the compensator method relative to the general integration algorithm that we shall present in next section let us consider the simplest example of non abelian maximally split coset, already studied and solved in [7]. This is the manifold

$$\mathbf{U}/\mathbf{H} = \text{SL}(3, \mathbb{R})/\text{SO}(3) \quad (2.37)$$

and the Lie algebra $\mathbf{U} = \text{SL}(3, \mathbb{R})$ is the maximally non-compact real form of the simple complex Lie algebra A_2 . The coset decomposition (2.3) takes the form:

$$\text{SL}(3, \mathbb{R}) = \text{SO}(3) \oplus \mathbb{K} \quad (2.38)$$

where

$$\begin{aligned} \text{SL}(3, \mathbb{R}) &= \text{Span}\{\mathcal{H}_i, E^\alpha, E^{-\alpha}\}, \quad i = 1, 2, \quad \alpha = 1, 2, 3, \\ \mathbb{K} &= \text{Span}\{\mathbb{K}_A\} = \text{Span}\left\{\mathcal{H}_i, \frac{1}{\sqrt{2}} (E^\alpha + E^{-\alpha})\right\}, \\ \text{SO}(3) &= \text{Span}\{t_\alpha\} = \text{Span}\{(E^\alpha - E^{-\alpha})\}. \end{aligned} \quad (2.39)$$

In the fundamental 3×3 representation the generators of $\text{SL}(3, \mathbb{R})$ are given by:

$$\begin{aligned} \mathcal{H}_1 &= \frac{1}{\sqrt{2}} \begin{pmatrix} 1 & 0 & 0 \\ 0 & -1 & 0 \\ 0 & 0 & 0 \end{pmatrix}, & \mathcal{H}_2 &= \frac{1}{\sqrt{6}} \begin{pmatrix} 1 & 0 & 0 \\ 0 & 1 & 0 \\ 0 & 0 & -2 \end{pmatrix}, \\ E^1 &= \begin{pmatrix} 0 & 1 & 0 \\ 0 & 0 & 0 \\ 0 & 0 & 0 \end{pmatrix}, & E^2 &= \begin{pmatrix} 0 & 0 & 0 \\ 0 & 0 & 1 \\ 0 & 0 & 0 \end{pmatrix}, & E^3 &= \begin{pmatrix} 0 & 0 & 1 \\ 0 & 0 & 0 \\ 0 & 0 & 0 \end{pmatrix}, \\ E^{-1} &= \begin{pmatrix} 0 & 0 & 0 \\ 1 & 0 & 0 \\ 0 & 0 & 0 \end{pmatrix}, & E^{-2} &= \begin{pmatrix} 0 & 0 & 0 \\ 0 & 0 & 0 \\ 0 & 1 & 0 \end{pmatrix}, & E^{-3} &= \begin{pmatrix} 0 & 0 & 0 \\ 0 & 0 & 0 \\ 1 & 0 & 0 \end{pmatrix}. \end{aligned} \quad (2.40)$$

In [7] we considered the differential system (2.35) generated by taking as generating solution in normal form the following:

$$\begin{aligned} \mathbf{Y}_{nf} &= (\nu_1, \nu_2, 0, 0, 0), \\ \nu_1 &= \frac{\omega - \kappa}{4\sqrt{2}} \quad ; \quad \nu_2 = \frac{3\omega + \kappa}{4\sqrt{6}} \end{aligned} \quad (2.41)$$

where ω and κ are just two alternative constant parameters replacing the components $\nu_{1,2}$ of the normal solution along the two Cartan generators. Parametrizing the $h \in \text{SO}(3)$ gauge group element as follows:

$$h = \exp[\theta_3(t) t_3] \exp[\theta_2(t) t_2] \exp[\theta_1(t) t_1] \quad (2.42)$$

we found in [7] that the differential system (2.35) takes the form

$$\begin{aligned} 0 &= \dot{\theta}_3(t) - \frac{1}{4}\omega \sin(2\theta_3(t)), \\ 0 &= 8\dot{\theta}_2(t) - [\kappa + \omega \cos(2\theta_3(t))] \sin(2\theta_2(t)), \\ 0 &= 16\dot{\theta}_1(t) + [\kappa + \kappa \cos(2\theta_2(t)) + \omega[\cos(2\theta_2(t)) - 3]\cos(2\theta_3(t))] \sin(2\theta_1(t)) - \\ &\quad - 8\omega \sin^2(\theta_1(t)) \sin(\theta_2(t)) \sin(2\theta_3(t)). \end{aligned} \quad (2.43)$$

This system has the requested triangular property and the first two equations are immediately solved by

$$\begin{aligned} \theta_3(t) &= -\arccos \left[-\frac{e^{\frac{\omega \lambda_3}{2}}}{\sqrt{e^{t\omega} + e^{\omega \lambda_3}}} \right], \\ \theta_2(t) &= -\arctan \sqrt{\frac{e^{\frac{t(\kappa+\omega)+\lambda_2}{2}}}{e^{t\omega} + e^{\omega \lambda_3}}} \end{aligned} \quad (2.44)$$

where $\lambda_{2,3}$ are two integration constants. The third equation of (2.43) was also solved in [7] with some ad hoc elaborations and in this way we obtained a general integral

depending on the five integration constants $\{\omega, \kappa, \lambda_1, \lambda_2, \lambda_3\}$. This is obviously consistent with the full integrability of the system demonstrated by the Lax pair representation, but still required some ad hoc elaborations. Furthermore for higher rank algebras like for instance A_3 , various experiments showed that although the system (2.35) can be arranged to triangular for a few rotations, it is not possible to make it completely triangular if we include rotations along all the generators of \mathcal{H} . These limitations are now completely overcome by the general integration formulae which we present in the next section.

3 The general integration algorithm.

For maximally split cosets \mathcal{U}/\mathcal{H} , choosing a solvable coset representative $\mathbb{L}(\phi)$, there is an established algorithm [17] to construct the general solution of eq. (2.16). For the maximally non compact real section of any simple Lie algebra \mathbb{U} there always exists a permutation matrix \mathcal{O} such that in the Weyl basis the Lax pair representation (2.16) can be brought to the following general form

$$\frac{d}{dt}L = [L, P] \quad (3.1)$$

where

$$L = \mathcal{O}V\mathcal{O}^T, \quad P = \mathcal{O}W\mathcal{O}^T \quad (3.2)$$

and the matrix P is given by the following projection of L

$$P = \Pi(L) := L_{>0} - L_{<0}, \quad (3.3)$$

$L_{>0}$ (<0) denoting the strictly upper (lower) triangular part of the $n \times n$ matrix L . The existence of the matrix \mathcal{O} , such that eq. (3.3) holds true is guaranteed by a theorem proved in Helgason's book [16], by the maximally split nature of the coset \mathcal{U}/\mathcal{H} and by the condition of solvability (2.33) imposed on the coset representative. The quoted theorem states that for *any linear representation of a solvable algebra* there always exist a basis where all generators are upper triangular matrices. Invoking this theorem for the solvable algebra $Solv(\mathcal{U}/\mathcal{H})$ generated by $\{\mathcal{H}_i, E^\alpha\}$ we conclude that we can always bring the semisimple operators \mathcal{H}_i to diagonal form and the nilpotent operators E^α to strictly upper triangular form. In the same basis the step-down operators $E^{-\alpha}$ which are the adjoints of the corresponding E^α are necessarily strictly lower triangular. Since the connection W is by definition given by $W = t_\alpha W^\alpha = W^\alpha (E^\alpha - E^{-\alpha})$, while V is given by $V = V^i \mathcal{H}_i + V^\alpha \frac{1}{\sqrt{2}} (E^\alpha - E^{-\alpha})$, we see that in the basis where E^α is upper triangular and \mathcal{H}_i diagonal, eq. (3.3) is nothing else but the solvability condition (2.33).

This established, we can proceed to apply the integration algorithm [17] in order to write the general integral of the Lax equation (3.1). Actually this is nothing else than an instance of the inverse scattering method. Indeed equation (3.1) represents the compatibility condition for the following linear system exhibiting the iso-spectral property of L

$$\begin{aligned} L\Psi &= \Psi\Lambda, \\ \frac{d}{dt}\Psi &= -P\Psi \end{aligned} \quad (3.4)$$

where $\Psi(t)$ is the eigenmatrix, namely the matrix whose i -th row is the eigenvector $\varphi(t, \lambda_i)$ corresponding to the eigenvalue λ_i of the Lax operator $L(t)$ at time t and Λ is the diagonal matrix of eigenvalues, which are constant throughout the whole time flow:

$$\begin{aligned}\Psi &= [\varphi(\lambda_1), \dots, \varphi(\lambda_n)] \equiv [\varphi_i(\lambda_j)]_{1 \leq i, j \leq n}, \\ \Psi^{-1} &= [\psi(\lambda_1), \dots, \psi(\lambda_n)]^T \equiv [\psi_j(\lambda_i)]_{1 \leq i, j \leq n}, \\ \Lambda &= \text{diag}(\lambda_1, \dots, \lambda_n).\end{aligned}\tag{3.5}$$

The solution of (3.4) for the Lax operator is given by the following explicit form of the matrix elements:

$$[L(t)]_{ij} = \sum_{k=1}^n \lambda_k \varphi_i(\lambda_k, t) \psi_j(\lambda_k, t). \tag{3.6}$$

The eigenvectors of the Lax operator at each instant of time, which define the eigenmatrix $\Psi(t)$, and the columns of the its inverse $\Psi^{-1}(t)$, are expressed in closed form in terms of the initial data at some conventional instant of time, say at $t = 0$.

Explicitly we have:

$$\begin{aligned}\varphi_i(\lambda_j, t) &= \frac{e^{-\lambda_j t}}{\sqrt{D_i(t)D_{i-1}(t)}} \text{Det} \begin{pmatrix} c_{11} & \dots & c_{1,i-1} & \varphi_1^0(\lambda_j) \\ \vdots & \ddots & \vdots & \vdots \\ c_{i1} & \dots & c_{i,i-1} & \varphi_i^0(\lambda_j) \end{pmatrix}, \\ \psi_j(\lambda_i, t) &= \frac{e^{-\lambda_i t}}{\sqrt{D_j(t)D_{j-1}(t)}} \text{Det} \begin{pmatrix} c_{11} & \dots & c_{1,j} \\ \vdots & \ddots & \vdots \\ c_{j-1,1} & \dots & c_{j-1,j} \\ \psi_1^0(\lambda_i) & \dots & \psi_j^0(\lambda_i) \end{pmatrix}\end{aligned}\tag{3.7}$$

where the time dependent matrix $c_{ij}(t)$ is defined below:

$$c_{ij}(t) = \sum_{k=1}^N e^{-2\lambda_k t} \varphi_i^0(\lambda_k) \psi_j^0(\lambda_k), \tag{3.8}$$

and

$$\begin{aligned}\varphi_i^0(\lambda_k) &:= \varphi_i(\lambda_k, 0) \\ \psi_i^0(\lambda_k) &:= \psi_i(\lambda_k, 0)\end{aligned}\tag{3.9}$$

are the eigenvectors and their adjoints calculated at $t = 0$. These constant vectors constitute the initial data of the problem and provide the integration constants. Finally $D_k(t)$ denotes the determinant of the $k \times k$ matrix with entries $c_{ij}(t)$

$$D_k(t) = \text{Det} \left[\left(c_{ij}(t) \right)_{1 \leq i, j \leq k} \right]. \tag{3.10}$$

Note that $c_{ij}(0) = \delta_{ij}$ and $D_k(0) = 1$ as well as $D_0(t) := 1$.

3.1 Parametrization of the initial data

Let us now reconsider the integration formulae presented in the previous subsection keeping in mind that, at any instant of time t , the Lax operator $L(t)$ is an element of the Lie algebra \mathbb{U} lying in the orthogonal complement \mathbb{K} to the subalgebra \mathbb{H} . This Lie algebra element is expressed in the chosen n -dimensional representation. Diagonalizing $L(t)$ means nothing else than bringing it inside the Cartan subalgebra $\text{CSA} \subset \mathbb{K} \subset \mathbb{U}$, which can always be performed by conjugation with an element of the maximal compact subgroup. Hence at every instant of time there exists an element $h(t) \in \exp[\mathbb{H}]$ such that:

$$\exists h(t) \in \exp[\mathbb{H}] \setminus h^{-1}(t) L(t) h(t) = \mathcal{C}_0 \in \text{CSA}. \quad (3.11)$$

The $n \times n$ matrix \mathcal{C}_0 is by construction diagonal and has therefore on its diagonal the above mentioned constant eigenvalues λ_i . What are these? They are just the n -weights of the chosen n -dimensional representation evaluated on the Cartan element \mathcal{C}_0 . Calling \mathbf{w}_i the weights, we just have:

$$\lambda_i = \mathbf{w}_i(\mathcal{C}_0) = \mathbf{w}_i \cdot \vec{\nu} \quad (i = 1, \dots, n) \quad (3.12)$$

having denoted by the r -vector $\vec{\nu}$ the r components of the Cartan element \mathcal{C}_0 in a basis of the Cartan subalgebra:

$$\mathcal{C}_0 = \nu_i \mathcal{H}^i \quad i = 1, \dots, r = \text{rank}(\mathbb{U}). \quad (3.13)$$

Equation (3.11) is in particular true at the initial time $t = 0$, where there exists some constant $h_0 \in \exp[\mathbb{H}]$ such that

$$h_0^{-1} L(0) h_0 = \mathcal{C}_0 = \text{diag}(\lambda_1 \dots, \lambda_n). \quad (3.14)$$

At this point all items entering the solution algorithm described in the previous section obtain their proper group-theoretical interpretation. The diagonal eigenvalue matrix Λ is nothing else but the CSA element \mathcal{C}_0 evaluated in the chosen n -dimensional representation. The eigenmatrix $\Psi(0)$ which provides the initial data of the problem is nothing else but the H -group element h_0 . This explains what we anticipated in section 1.2 when we stated that the initial data are a pair of CSA algebra element and a H -subgroup element. We still have to verify the other properties relative to the Weyl group. These are also easily understood. By definition the Weyl group \mathcal{W} is that discrete subgroup of H which maps the Cartan subalgebra into itself:

$$\forall w \in \mathcal{W}, \forall \mathcal{C} \in \text{CSA} \quad w^{-1} \cdot \mathcal{C} \cdot w \in \text{CSA}. \quad (3.15)$$

Hence if we choose h_0 inside the Weyl group the initial value of the Lax operator $L_0 = L(0)$ remains a diagonal matrix in the Cartan subalgebra. If L_0 is diagonal the connection $P(0) = 0$ vanishes at the origin and from Lax equation we obtain

$$\frac{d}{dt} L|_{t=0} = 0 \quad (3.16)$$

which implies that the solution is constant throughout the whole flow: $L(t) = L(0)$. This suffices to prove eq. (1.6). On the other the so called sorting property, namely the statement (1.7) was proved by Kodama et al in [17]. As already anticipated in section 1.2 eq. (1.6), together with the iso-spectral property, namely with the form of the eigenvalues (3.12) implies that eq. (1.9) has also to be true. Finally the independence of the charge Ω from the eigenvalues, namely from the choice of the CSA element \mathcal{C}_0 , together with eq. (1.6) implies that the map induced by the solution algorithm is indeed of the type described in eq. (1.10).

4 The A_3 example with numerical plots

In order to exemplify the value and the use of the general integration algorithm, we will apply it to the case of the manifold

$$\mathcal{M}_3 \equiv \frac{\text{SL}(4, \mathbb{R})}{\text{O}(4)} \quad (4.1)$$

which is maximally split of rank $r = 3$ and corresponds to the Lie algebra A_3 . As in the previous A_2 case we can use the fundamental representation as working representation for the Lax operator, which means that we deal with 4×4 matrices. The dimension of the coset is 9 and there are 6 positive roots:

$$\begin{aligned} \alpha_1 &= \{\sqrt{2}, 0, 0\} & ; & \quad \alpha_2 = \left\{-\frac{1}{\sqrt{2}}, -\sqrt{\frac{3}{2}}, 0\right\}, \\ \alpha_3 &= \left\{0, \sqrt{\frac{2}{3}}, \frac{2}{\sqrt{3}}\right\} & ; & \quad \alpha_4 = \left\{\frac{1}{\sqrt{2}}, -\sqrt{\frac{3}{2}}, 0\right\}, \\ \alpha_5 &= \left\{-\frac{1}{\sqrt{2}}, -\frac{1}{\sqrt{6}}, \frac{2}{\sqrt{3}}\right\} & ; & \quad \alpha_6 = \left\{\frac{1}{\sqrt{2}}, -\frac{1}{\sqrt{6}}, \frac{2}{\sqrt{3}}\right\}. \end{aligned} \quad (4.2)$$

The matrices representing the Cartan generators well adapted to the representation (4.2) are the following ones:

$$\begin{aligned} \mathcal{H}_1 &= \begin{pmatrix} \frac{1}{\sqrt{2}} & 0 & 0 & 0 \\ 0 & -\frac{1}{\sqrt{2}} & 0 & 0 \\ 0 & 0 & 0 & 0 \\ 0 & 0 & 0 & 0 \end{pmatrix} & ; & \quad \mathcal{H}_2 = \begin{pmatrix} -\frac{1}{\sqrt{6}} & 0 & 0 & 0 \\ 0 & -\frac{1}{\sqrt{6}} & 0 & 0 \\ 0 & 0 & \sqrt{\frac{2}{3}} & 0 \\ 0 & 0 & 0 & 0 \end{pmatrix} ; \\ \mathcal{H}_3 &= \begin{pmatrix} \frac{1}{2\sqrt{3}} & 0 & 0 & 0 \\ 0 & \frac{1}{2\sqrt{3}} & 0 & 0 \\ 0 & 0 & \frac{1}{2\sqrt{3}} & 0 \\ 0 & 0 & 0 & \frac{-\sqrt{3}}{2} \end{pmatrix} ; \end{aligned} \quad (4.3)$$

while the 6 step-up operators associated with the positive roots are given by:

$$E^{\alpha_1} = \begin{pmatrix} 0 & 1 & 0 & 0 \\ 0 & 0 & 0 & 0 \\ 0 & 0 & 0 & 0 \\ 0 & 0 & 0 & 0 \end{pmatrix} ; \quad E^{\alpha_2} = \begin{pmatrix} 0 & 0 & 0 & 0 \\ 0 & 0 & 1 & 0 \\ 0 & 0 & 0 & 0 \\ 0 & 0 & 0 & 0 \end{pmatrix}, \quad (4.4)$$

$$E^{\alpha_3} = \begin{pmatrix} 0 & 0 & 0 & 0 \\ 0 & 0 & 0 & 0 \\ 0 & 0 & 0 & 1 \\ 0 & 0 & 0 & 0 \end{pmatrix} ; \quad E^{\alpha_4} = \begin{pmatrix} 0 & 0 & 1 & 0 \\ 0 & 0 & 0 & 0 \\ 0 & 0 & 0 & 0 \\ 0 & 0 & 0 & 0 \end{pmatrix}, \quad (4.5)$$

$$E^{\alpha_5} = \begin{pmatrix} 0 & 0 & 0 & 0 \\ 0 & 0 & 0 & 1 \\ 0 & 0 & 0 & 0 \\ 0 & 0 & 0 & 0 \end{pmatrix} ; \quad E^{\alpha_6} = \begin{pmatrix} 0 & 0 & 0 & 1 \\ 0 & 0 & 0 & 0 \\ 0 & 0 & 0 & 0 \\ 0 & 0 & 0 & 0 \end{pmatrix}. \quad (4.6)$$

The step-down operators associated with the negative roots are just the transposed of the corresponding step-up operators.

The H subgroup is $O(4)$, namely the set of all orthogonal 4×4 -matrices. The Weyl group $\mathcal{W} \subset O(4) \subset \text{SL}(4, \mathbb{R})$ is S_4 and its elements are represented by the orthogonal matrices associated with the $4!$ permutations $P \in S_4$ in the following way:

$$\begin{aligned} (O_P)_{i,P(i)} &= 1, \\ (O_P)_{ij} &= 0 \quad \text{otherwise.} \end{aligned} \quad (4.7)$$

So for instance the cyclic permutation

$$P = \begin{array}{cccc} 1 & 2 & 3 & 4 \\ \downarrow & \downarrow & \downarrow & \downarrow \\ 4 & 1 & 2 & 3 \end{array} \quad (4.8)$$

is represented by the matrix

$$O_P = \begin{pmatrix} 0 & 0 & 0 & 1 \\ 1 & 0 & 0 & 0 \\ 0 & 1 & 0 & 0 \\ 0 & 0 & 1 & 0 \end{pmatrix}. \quad (4.9)$$

Having defined the model we can now study the solutions produced by the general integration algorithm implementing it on a computer and choosing numerical values for the initial data.

Let us begin with the choice of the Cartan subalgebra element \mathcal{C}_0 . Setting:

$$\nu_1 = -\left(\frac{\lambda_1 + \lambda_2 + 2\lambda_3}{\sqrt{2}}\right) ; \quad \nu_2 = \frac{\lambda_1 + 3\lambda_2}{\sqrt{6}} ; \quad \nu_3 = \frac{-2\lambda_1}{\sqrt{3}}. \quad (4.10)$$

We obtain:

$$\mathcal{C}_0 \equiv \sum_{i=1}^3 \nu_i \mathcal{H}_i = \begin{pmatrix} -\lambda_1 - \lambda_2 - \lambda_3 & 0 & 0 & 0 \\ 0 & \lambda_3 & 0 & 0 \\ 0 & 0 & \lambda_2 & 0 \\ 0 & 0 & 0 & \lambda_1 \end{pmatrix}. \quad (4.11)$$

We are free to choose the eigenvalues λ_i . We exhibit the solution for the simple choice:

$$\lambda_i = i \quad \Rightarrow \quad \mathcal{C}_0 = \begin{pmatrix} -6 & 0 & 0 & 0 \\ 0 & 3 & 0 & 0 \\ 0 & 0 & 2 & 0 \\ 0 & 0 & 0 & 1 \end{pmatrix}. \quad (4.12)$$

Next we have to choose the H -subgroup element. Our numerical choice is given by the following orthogonal matrix:

$$h_0 = \begin{pmatrix} \frac{-3-\sqrt{3}}{8} & \frac{-5(-1+\sqrt{3})}{16} & \frac{1+\sqrt{3}}{4\sqrt{2}} & \frac{-1-5\sqrt{3}}{16} \\ \frac{-(-1+\sqrt{3})}{4\sqrt{2}} & \frac{\sqrt{2}+3\sqrt{6}}{16} & \frac{1+\sqrt{3}}{4} & \frac{-(-7+\sqrt{3})}{8\sqrt{2}} \\ \frac{-(3+2\sqrt{3})}{8\sqrt{2}} & \frac{7+4\sqrt{3}}{16\sqrt{2}} & \frac{-6+\sqrt{3}}{8} & \frac{-4+\sqrt{3}}{16\sqrt{2}} \\ \frac{-8+\sqrt{3}}{8\sqrt{2}} & \frac{-(10+\sqrt{3})}{16\sqrt{2}} & -\left(\frac{1}{8}\right) & \frac{11+2\sqrt{3}}{16\sqrt{2}} \end{pmatrix} \quad (4.13)$$

which is just a product of rotations along the six generators of $O(4)$ with angles that are multiples of $\pi/3$ or $\pi/4$ in order to obtain simple but non trivial entries all over the place. With such initial data the computer programme calculates the Lax operator $L(t)$ at all times and finds that the limits at $\pm\infty$ are as follows:

$$\begin{aligned} \lim_{t \rightarrow -\infty} L(t) &= \begin{pmatrix} 3 & 0 & 0 & 0 \\ 0 & 2 & 0 & 0 \\ 0 & 0 & 1 & 0 \\ 0 & 0 & 0 & -6 \end{pmatrix}, \\ \lim_{t \rightarrow \infty} L(t) &= \begin{pmatrix} -6 & 0 & 0 & 0 \\ 0 & 1 & 0 & 0 \\ 0 & 0 & 2 & 0 \\ 0 & 0 & 0 & 3 \end{pmatrix}. \end{aligned} \quad (4.14)$$

This means that the image of $h_0 \in O(4)$ under the map \mathcal{P}_K of eq. (1.10) is the permutation displayed below:

$$\mathcal{P}_K : \frac{O(4)}{S_4} \ni h_0 \rightarrow \pi_1(h_0) = \begin{pmatrix} 0 & 0 & 0 & 1 \\ 0 & 0 & 1 & 0 \\ 0 & 1 & 0 & 0 \\ 1 & 0 & 0 & 0 \end{pmatrix} \in S_4. \quad (4.15)$$

The permutation of the eigenvalues occurs through a smooth process that realizes the billiard bounces and that can be appreciated through the inspection of plots of the Cartan fields. The most instructive plots are those of the differential and integrated Cartan fields. Given the explicit solution for the Lax operator $L(t)$ obtained through implementation of the integration algorithm, let us define:

$$\vec{\chi}_x(t) \equiv \text{Tr}[\mathcal{H}_x L(t)] \quad (4.16)$$

the components of the tangent vector along the geodesic. Let us also define the projection

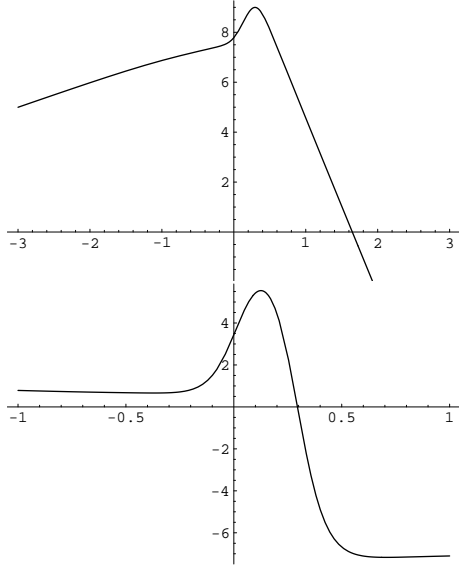


Figure 1: Plots of the differential and integrated Cartan fields $\chi_{\alpha_1}(t)$ and $h_{\alpha_1}(t)$, along the first simple root of the A_3 Lie algebra and for the solution generated by the initial data C_0 and h_0 , given in eqs. (4.12) and (4.13), respectively. The plot on the left is $h_{\alpha_1}(t)$, while the plot on the right is $\chi_{\alpha_1}(t)$. We see the bounce at $t = 0.29488$.

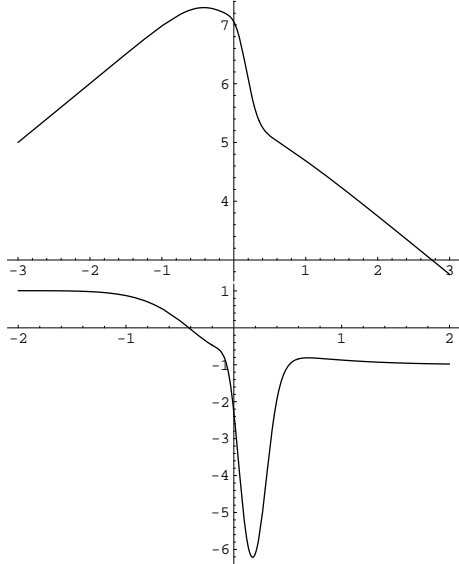


Figure 2: Plots of the differential and integrated Cartan fields $\chi_{\alpha_2}(t)$ and $h_{\alpha_2}(t)$, along the second simple root of the A_3 Lie algebra and for the solution generated by the initial data C_0 and h_0 , given in eqs. (4.12) and (4.13), respectively. The plot on the left is $h_{\alpha_2}(t)$, while the plot on the right is $\chi_{\alpha_2}(t)$. We see the bounce at $t = -0.416$.

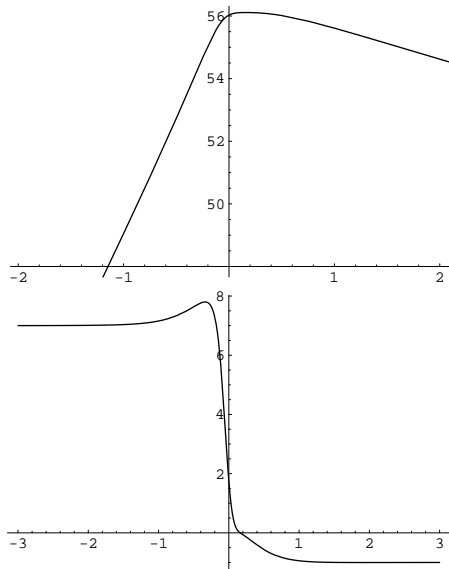


Figure 3: Plots of the differential and integrated Cartan fields $\chi_{\alpha_3}(t)$ and $h_{\alpha_3}(t)$, along the third simple root of the A_3 Lie algebra and for the solution generated by the initial data C_0 and h_0 , given in eqs. (4.12) and (4.13), respectively. The plot on the left is $h_{\alpha_3}(t)$, while the plot on the right is $\chi_{\alpha_3}(t)$. We see the bounce at $t = 0.160124$.

of this tangent vector along the simple roots

$$\chi_{\alpha_i}(t) = \vec{\chi}(t) \cdot \alpha_i \quad ; \quad i = 1, 2, 3 . \quad (4.17)$$

Finally the Cartan fields, that in supergravity correspond to the cosmological scale factors and to the dilatons ¹, are defined by simple integration:

$$h_{\alpha_i}(t) \equiv \int_{t_0}^t \chi_{\alpha_i}(\ell) d\ell \quad ; \quad i = 1, 2, 3 . \quad (4.18)$$

In fig.s 1,2,3 we present, for the considered explicit solution the plots of $\chi_{\alpha_i}(t)$ and $h_{\alpha_i}(t)$. Billiard bounces correspond to maxima or minima of the simple root integrated Cartan fields $h_{\alpha_i}(t)$ or, equivalently to zeros of the differential ones $\chi_{\alpha_i}(t)$. Inspection of the plots reveals that the considered solution, characterized by the topological charge of equation (4.15) admits three bounces, respectively located at:

$$t = 0.160124 \quad ; \quad t = -0.416 \quad ; \quad t = 0.29488 . \quad (4.19)$$

5 Billiard interpretation

Having in mind the explicit example we have just illustrated we can now emphasize the physical *billiard interpretation* of the mathematical results we obtained.

¹See [7, 8] for detailed explanations about the oxidation mechanism that lifts the σ -model solutions to full-fledged solutions of supergravity theory in higher dimensions. See also [10] and [9] for the relation between the σ -model and $D = 4$ supergravity with all number of supercharges.

At those times when the Lax operator $L(t)$ lies in the Cartan subalgebra, the fictitious cosmic ball moves on a straight line, the components of $L(t)$ along the CSA generators \mathcal{H}_x being the components of its velocity:

$$\overrightarrow{v}_x(t) = \text{Tr} [\mathcal{H}_x L(t)]. \quad (5.1)$$

This means that the multidimensional Universe has a constant rate (positive or negative) exponential expansion (contraction) in the corresponding dimensions. The asymptotic theorem states that for finite Lie algebras, namely as long as we confine our attention to time-dependent backgrounds that can be retrieved by oxidation from $D = 3$ supergravity (see [7, 8, 9, 10]), the Universe is in such a *constant velocity state* at asymptotically early and asymptotically late times. Furthermore given the initial state:

$$|\overrightarrow{v}, -\infty > \quad (5.2)$$

at $t = -\infty$ (Big Bang time) the entire cosmic evolution can be seen as a smooth process that brings the Universe to another such state:

$$|\overrightarrow{v}', +\infty > . \quad (5.3)$$

The surprising result is that there is a finite countable numbers of such possible states, as many as the elements of the Weyl group \mathcal{W} . Indeed the final Universe velocity is necessarily the image of its initial one under the action of some element of \mathcal{W} . The transition from one state to the other occurs through one or several bounces on the dynamical walls of the Weyl chamber that as we have already emphasized in previous papers raise and decay smoothly.

6 Conclusions and Perspectives

We could now show explicit analytic formulae for the general integral of simple models like the A_2 model. It is very easy to obtain them by means of computer codes written in MATHEMATICA but they are very lengthy and clumsy to be displayed so that we prefer not to. The relevant point is that the integration algorithm is fully explicit and we hope we have been able to illustrate this fact. This enables us to apply our method to any case of interest, as long as the target manifold is maximally split. We can also generate, by means of the *paint group rotations* entire classes of solutions in the non maximally split case. However they do not yet constitute a general integral in these cases. The open problem is therefore that of extending the results presented in this article in the following directions:

1. For the non maximally split non compact cosets appearing in supergravity with non maximal SUSY.
2. For the infinite dimensional Kač–Moody algebras generated by reduction to $D = 2$ and $D = 1$. This is particularly important in view of the results we have achieved on asymptotic regimes. It seems quite clear that the chaotic Kasner behaviour

predicted around initial or final singularities must be related to the essential change in structure of the Weyl group for Kač–Moody algebras. The presence of infinitely many roots and moreover the presence of space-like or null-like roots makes the Weyl group much more complicated and apt to create smooth solutions with singular behaviour if there is an extension of our main results to the Kač–Moody case.

3. For σ -models deformed by the presence of a potential, like it happens when fluxes are turned on and one considers gauged supergravity rather than ungauged supergravity.
4. Generalization of the results obtained in this paper to the case of one-dimensional super-coset sigma models, which comprise both bosonic and fermionic fields.

On all of these four issues we are presently working and hope to be able to present new conclusions soon.

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